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Hadron collisions and the fifth form factor

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Abstract

Logarithmically enhanced effects due to radiation of soft gluons at large angles in $2 \to 2$ QCD scattering processes are treated in terms of the "fifth form factor" that accompanies the four collinear singular Sudakov form factors attached to incoming and outgoing hard partons. Unexpected symmetry under exchange of internal and external variables of the problem is pointed out for the anomalous dimension that governs soft gluon effects in hard gluon–gluon scattering.

1 Cross-channel colour transfer and soft gluons

Measuring final–state characteristics in hard hadron–hadron interactions supplements the overall hardness scale Q of the underlying parton scattering process $p_1, p_2 \to p_3, p_4$ with the second (hard) scale $Q_0 \ll Q$ that quantifies small deviation of the final state system from the Born kinematics (out-of-event-plane particle production, near-to-backward particle correlations, inter-jet energy flows, etc.). The ratio of these two scales being a large parameter calls for analysis and resummation of double (DL) and single logarithmic (SL) radiative corrections in all orders. Logarithmically enhanced (both DL and SL) contributions of *collinear* origin are easy to analyse and resum into exponential Sudakov form factors belonging to hard primary partons.

SL effects due to soft gluons radiated at large angles pose more problems. For example, particle energy flow $E = Q_0$ in a given inter-jet direction acquires contributions $\mathcal{O}\left(\alpha_s^n \ln^n(Q/Q_0)\right)$ from ensembles of n energy ordered gluons radiated at arbitrary (large) angles. In general, such "hedgehog" multi-gluon configurations contribute at the SL level to the so called *non-global* observables that acquire contributions from a restricted phase space window [1] and are difficult to analyse. On the contrary, *global observables* that acquire contributions from the full phase space are free from this trouble: only the hardest among the secondary gluons contributes while the softer ones don't affect the observable so that their contributions cancel against corresponding virtual terms. As a result, the problem reduces to the analysis of *virtual corrections* due to multiple gluons with $k_t > Q_0$ attached to primary hard partons only. They can be treated iteratively and fully exponentiated, together with DL terms.

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The programme of resumming soft SL effects due to large angle gluon emission in hadron–hadron collisions was pioneered by Botts and Sterman [2] (see also [3–5]). Complication arises from the fact that gluon emission changes the *colour state* of the hard parton system which in turn affects successive radiation of a softer gluon.

In this letter we propose a general treatment of large angle gluon radiation. It is based on the observation that the square of the eikonal current for emission of gluon $k=(\omega,\mathbf{k})$ off an ensemble of four partons $i=1,\dots 4$,

$$j^{\mu,b}(k) = \sum_{i=1}^{4} \frac{\omega\, p_i^{\mu}}{(kp_i)} T_i^b\,; \qquad \sum_{i=1}^{4} T_i^b \;=\; 0, \tag{1}$$

can be represented as

$$\begin{aligned} -j^2(k) = {} & T_1^2\, W_{34}^{(1)}(k) + T_2^2\, W_{34}^{(2)}(k) + T_3^2\, W_{12}^{(3)}(k) + T_4^2\, W_{12}^{(4)}(k) \\ & + \;T_t^2\cdot A_t(k) \;\; + T_u^2 \cdot A_u(k)\,. \end{aligned} \tag{2}$$

Here T_i^2 is the $SU(N)$ "colour charge" of parton p_i and the two operators T_t^2 and T_u^2 are the charges exchanged in the t- and u- channels of the scattering process,

$$T_t^2 = (T_3+T_1)^2 = (T_2+T_4)^2, \quad T_u^2 = (T_4+T_1)^2 = (T_2+T_3)^2. \tag{3}$$

The functions W are combinations of *dipole antennae*

$$W_{34}^{(1)} = w_{13} + w_{14} - w_{34}\,, \quad w_{ij}(k) = \frac{\omega^2\,(p_ip_j)}{(kp_i)(kp_j)}. \tag{4}$$

The distribution (4) is collinear singular *only* when $\mathbf{k} \parallel \mathbf{p}_1$. This singularity contributes proportional to the corresponding Casimir operator, in accord with general factorisation property. Integrating (4) over angles gives

$$\int \frac{d\Omega}{4\pi}\, W_{34}^{(1)} \;=\; \ln \frac{(p_1p_3)(p_1p_4)}{(p_3p_4)\, m^2} = \ln \frac{tu}{2m^2\, s}\,. \tag{5}$$

with m^2 the collinear cutoff and $s=(p_1{+}p_2)^2$, $t=(p_1{-}p_3)^2$, $u=(p_1{-}p_4)^2$. The collinear cutoff m disappears when the virtual and real contributions (to a collinear and infrared safe observable) are taken together, and gets replaced by the proper observable dependent scale Q_0, see examples in [6].

The first four terms in (2) are collinear singular and belong to individual partons; their colour factors are numbers. Their exponentiation leads to the product of four DL Sudakov form factors $F_i(Q_0,Q)$ with the common hard scale $Q^2 = tu/s = s\sin^2\Theta_s$.

The angular dipole combinations in the last two terms in (2) are

$$A_t = w_{12} + w_{34} - w_{13} - w_{24}\,, \quad A_u = w_{12} + w_{34} - w_{14} - w_{23}\,. \tag{6}$$

Unlike the dipoles $W_{jk}^{(i)}$, they are integrable in angles:

$$\int \frac{d\Omega}{4\pi}\, A_t(k) \;=\; 2\ln\frac{s}{-t}\,; \qquad \int \frac{d\Omega}{4\pi}\, A_u(k) \;=\; 2\ln\frac{s}{-u}\,. \tag{7}$$

Contrary to the first four DL contributions, this additional contribution originates from coherent gluon radiation at angles *larger than the cms scattering angle* Θ_s and gives rise to the *fifth form factor* $F_X(\tau)$ which is a single logarithmic function of the parameter

$$\tau = \int_{Q_0}^{Q} \frac{dk_t}{k_t} \frac{\alpha_s(k_t)}{\pi} \,. \tag{8}$$

Virtual corrections to the hard scattering matrix element are calculated simply by exponentiating a minus half of the real eikonal emission probability (2). Additional Coulomb corrections due to virtual gluon exchanges between two incoming or two outgoing partons are obtained from (2) by keeping only s-channel interference terms w_{12} and w_{34} and replacing them by $i\pi$. The finite non-Abelian Coulomb phase[2] can be simply incorporated by adding the phases to the logarithms in (7).

In summary, virtual dressing of the hard matrix element results in

$$M_0 \implies \prod_{i=1}^{4} F_i(Q_0, Q) \cdot M(\tau), \qquad M(\tau) = F_X(\tau) \cdot M_0 \,, \tag{9}$$

where F_X (with subscript X standing for "cross-channel") is given by

$$F_X(\tau) \;=\; \exp\left\{-\tau\left(T_t^2 \cdot T + T_u^2 \cdot U\right)\right\}, \quad T = \ln\frac{s}{t} = \ln\frac{s}{-t} - i\pi, \;\; U = \ln\frac{s}{u} \,. \tag{10}$$

This expression for the fifth form factor holds for scattering of arbitrary colour objects. It is present in the QED context as well (where it is determined by electric charge transfers in cross channels), however in QCD the operators T_t^2 and T_u^2 do not commute and this is what complicates the analysis.

The two scale distributions acquire the *soft factor* $\mathcal{S}_X$:

$$\Sigma(Q_0, Q) \;=\; \Sigma^{\rm coll}(Q_0, Q) \cdot \mathcal{S}_X(\tau), \qquad \mathcal{S}_X(\tau) \;=\; \frac{\mathrm{Tr}(M^\dagger(\tau)\, M(\tau))}{\mathrm{Tr}(M_0^\dagger\, M_0)} \cong |F_X|^2 \,. \tag{11}$$

Here $\Sigma^{\rm coll}$ embodies the first four terms in (2) as well as collinear logarithms from parton distribution functions.

It is convenient to work in the colour basis of s-channel projectors $\mathcal{P}_\alpha$ onto the irreducible $SU(N)$ representations α present in the colour space of the two incoming partons (p_1, p_2). The colour transfer matrices T_t^2 and T_u^2 in (10) can be easily found with use of the re-projection matrices[3] that express t- and u-channel colour projectors in terms of s-channel ones, $\mathcal{P}^{(t)} = K_{ts} \cdot \mathcal{P}$, $\mathcal{P}^{(u)} = K_{us} \cdot \mathcal{P}$, and vice versa, $K_{st} = K_{ts}^{-1}$, $K_{su} = K_{us}^{-1}$. We have

$$T_t^2 \;=\; K_{st}\, C_2^{(t)}\, K_{ts}, \quad T_u^2 \;=\; K_{su}\, C_2^{(u)}\, K_{us}, \tag{12}$$

with C_2 the diagonal matrix of Casimir operators of all irreducible representations present in the t (u) channel. An advantage of the representation of the *soft anomalous dimension*

$$\Gamma \;=\; -\left(T_t^2 \cdot T + T_u^2 \cdot U\right) \;\equiv -N(T+U) \cdot \mathcal{Q} \tag{13}$$

[2]a divergent part of the Coulomb phase, though non-Abelian, factors out and cancels in observables [7]

[3]For $gg \to gg$ these matrices are given in Appendix, see (33).

in terms of cross-channel charges (12) is trivialisation of the analysis of the Regge behaviour. In the case of small angle scattering one term dominates, $T \gg U$ (forward scattering) or $U \gg T$ (backward), and the anomalous dimension Γ becomes diagonal in the corresponding channel so that the problem becomes essentially Abelian. Resulting exponents are nothing but Regge trajectories of t-(u-) channel exchanges that are proportional to corresponding Casimirs.

2 Gluon–gluon scattering

As an example we considered in detail the case of gluon–gluon scattering that was first treated by Kidonakis, Oderda and Sterman in [3] and turned out to be rather complicated since it involves many colour channels (six for $SU(N)$).

Characterised in terms of irreducible representations, two gluons in $SU(3)$ can be in the states

$$\mathbf{glue} \otimes \mathbf{glue} = \mathbf{8_a} + \mathbf{10} + \mathbf{1} + \mathbf{8_s} + \mathbf{27}, \tag{14}$$

where $\mathbf{8_a}$ and $\mathbf{10}$ mark *antisymmetric* representations (with corresponding dimensions) and three *symmetric* ones are the singlet ($\mathbf{1}$), octet ($\mathbf{8_s}$) and the high symmetric tensor representation ($\mathbf{27}$). In the general case of $SU(N)$ (with $N > 3$) we have an additional *symmetric* representation (which we mark $\mathbf{0}$):

$$\mathbf{glue} \otimes \mathbf{glue} = \mathbf{8_a} + \mathbf{10} + \mathbf{1} + \mathbf{8_s} + \mathbf{27} + \mathbf{0}. \tag{15}$$

We use the $SU(3)$ motivated names in spite of the fact that the dimensions of corresponding representations are actually different from 8, 10, etc.:

$$\begin{aligned} K_{\mathbf{1}} = 1, \qquad K_{\mathbf{a}} = K_{\mathbf{s}} = N^2 - 1, \qquad K_{\mathbf{10}} &= 2 \cdot \frac{(N^2-1)(N^2-4)}{4}, \\ K_{\mathbf{27}} = \frac{N^2(N-1)(N+3)}{4}, \qquad\qquad K_{\mathbf{0}} &= \frac{N^2(N+1)(N-3)}{4}. \end{aligned} \tag{16}$$

The s-channel projectors $\mathcal{P}_\alpha$ are explicitly constructed in [7]. The projector basis we order as follows:

$$\mathcal{P}_\alpha = \{\mathcal{P}_{\mathbf{a}},\, \mathcal{P}_{\mathbf{10}}\,, \mathcal{P}_{\mathbf{1}},\, \mathcal{P}_{\mathbf{s}},\, \mathcal{P}_{\mathbf{27}},\, \mathcal{P}_{\mathbf{0}}\}\,. \tag{17}$$

The corresponding Casimir operators read

$$(T^a)^2_{\alpha\beta} \;=\; (C_2)_{\alpha\beta} = \delta_{\alpha\beta} \cdot c_\alpha\,, \qquad c_\alpha = \{N,\, 2N,\, 0,\, N,\, 2(N{+}1),\, 2(N{-}1)\}. \tag{18}$$

where C_2 is the diagonal matrix entering (12).

The normalised anomalous dimension $\mathcal{Q}$ defined in (13) is a 6×6 matrix that depends on N and the ratio of logarithms T/U as unique geometry dependent (complex) parameter. It has six eigenstates. Three simple, N-independent, eigenvalues are

$$E_1 = 1\,, \quad E_2 = \frac{3-b}{2}\,, \quad E_3 = \frac{3+b}{2}\,; \qquad b \equiv \frac{T-U}{T+U}\,. \tag{19}$$

Three N-dependent eigenvalues satisfy the cubic equation

$$\left[E_i - \frac{4}{3}\right]^3 - \frac{(1+3b^2)(1+3x^2)}{3}\left[E_i - \frac{4}{3}\right] - \frac{2(1-9b^2)(1-9x^2)}{27} \;=\; 0; \qquad x \equiv \frac{1}{N}\,. \tag{20}$$

Its solutions can be parametrised as follows:

$$E_{4,5,6} \;=\; \frac{4}{3}\left(1+\frac{\sqrt{(1+3b^2)(1+3x^2)}}{2}\,\cos\left[\frac{\phi+2k\pi}{3}\right]\right), \qquad k=0,\,1,\,2\,; \tag{21a}$$

$$\cos\phi \;=\; R\,, \qquad R=\frac{(1-9b^2)(1-9x^2)}{\left[(1+3b^2)(1+3x^2)\right]^{3/2}}\,. \tag{21b}$$

In our representation the three N-dependent energy levels (21) and corresponding eigenvectors are explicitly real functions of b (the property not easy to extract from [3]). We give explicit expression for the soft factor $\mathcal{S}_X$ in special cases when eigenvalues (21) simplify.

2.1 Scattering at 90^o

The simplest case is 90 degree scattering which corresponds to $b{=}0$ ($t=u$). Here $\mathcal{Q}$ (32) is diagonal so that the s-channel projectors $\mathcal{P}_\alpha$ become eigenvectors whose eigenvalues are just the corresponding diagonal elements of $\mathcal{Q}$:

$$E_k \;=\; \left\{1,\,\frac{3}{2},\,\frac{3}{2},\,2,\,\frac{N-1}{N},\,\frac{N+1}{N}\right\}, \tag{22}$$

$$\mathcal{V}_\kappa \;\propto\; \{\mathcal{P}_{\mathbf{10}},\,\mathcal{P}_{\mathbf{s}}+\mathcal{P}_a,\,\mathcal{P}_{\mathbf{s}}-\mathcal{P}_a,\,\mathcal{P}_{\mathbf{1}},\,\mathcal{P}_{\mathbf{27}},\,\mathcal{P}_{\mathbf{0}}\}\,. \tag{23}$$

To present the answer for the soft factor $\mathcal{S}_X$ we define the suppression factors

$$\chi_t(\tau) \;=\; \exp\left\{-2N\tau\cdot\ln\frac{s}{-t}\right\}, \quad \chi_u(\tau) \;=\; \exp\left\{-2N\tau\cdot\ln\frac{s}{-u}\right\}. \tag{24}$$

In 90^o scattering kinematics we have $\chi_t=\chi_u$, $\mathrm{Re}\,T=\mathrm{Re}\,U=\ln 2$ and we get

$$\mathcal{S}_X(\tau)=\frac{\chi^2}{3}\left[\frac{4\,\chi^2}{N^2-1}+\chi+\frac{N-3}{N-1}\,\chi^{\frac{2}{N}}+\frac{N+3}{N+1}\,\chi^{-\frac{2}{N}}\right], \qquad \chi(\tau) \;=\; \exp\{-2N\tau\ln 2\}\,. \tag{25}$$

2.2 $N\to\infty$ limit

Now the eigenvalues of $\mathcal{Q}$ are as follows:

$$E_1=1, \quad E_2=\frac{3-b}{2}, \quad E_3=\frac{3+b}{2}, \quad E_4=2, \quad E_5=1-b, \quad E_6=1+b\,,$$

and the corresponding eigenvectors are

$$\mathcal{V}_{1,\ldots 6} \;=\; \begin{bmatrix}0\\0\\0\\0\\1\\-1\end{bmatrix}\begin{bmatrix}1+b\\-2b\\0\\1+b\\-b\\-b\end{bmatrix}\begin{bmatrix}1-b\\2b\\0\\-1+b\\-b\\-b\end{bmatrix}\begin{bmatrix}-4(1-b^2)\\-8b^3\\(1-b^2)^2\\4(1-b^2)\\2b^2(1+b^2)\\2b^2(1+b^2)\end{bmatrix}\begin{bmatrix}0\\2\\0\\0\\1\\1\end{bmatrix}\begin{bmatrix}0\\-2\\0\\0\\1\\1\end{bmatrix}. \tag{26}$$

The soft factor becomes

$$\mathcal{S}_X(\tau) \;=\; \chi_t\,\chi_u\,\frac{(m_t+m_u)^2 + (m_s-m_u)^2\,\chi_t + (m_s+m_t)^2\,\chi_u}{(m_t+m_u)^2 + (m_s-m_u)^2 + (m_s+m_t)^2}. \tag{27}$$

Here m_s, m_t, m_u are pieces of the Born gg scattering matrix element each containing the gluon exchange diagram in the corresponding channel (together with the piece of the four-gluon vertex with the same colour structure). The combinations m_t+m_u, m_s+m_t and m_s-m_u are gauge invariant amplitudes,

$$(m_t+m_u)^2 \;=\; 1-\frac{st}{u^2}-\frac{us}{t^2}+\frac{s^2}{tu},$$

with the other two obtained by crossing symmetry. The squared matrix element reads

$$\mathrm{Tr}(M_0^\dagger\, M_0) \;=\; \tfrac{1}{2}N^2(N^2-1)\left[\,(m_t+m_u)^2+(m_s-m_u)^2+(m_s+m_t)^2\right].$$

2.3 Regge limit

In the case of close to forward scattering, $|t| \ll s \simeq |u|$, we have $b \to 1$ at which point eigenvalues (21a) also simplify. Since U in (13) is negligible, according to (12) the eigenvectors coincide with t-channel projectors,

$$\begin{array}{lll} \mathcal{V}_1 = K_{st}\cdot\mathcal{P}^{(t)}_{\mathbf{a}}, & \mathcal{V}_2 = K_{st}\cdot\mathcal{P}^{(t)}_{\mathbf{s}}, & \mathcal{V}_3 = K_{st}\cdot\mathcal{P}^{(t)}_{\mathbf{10}}, \\ \mathcal{V}_4 = K_{st}\cdot\mathcal{P}^{(t)}_{\mathbf{1}}, & \mathcal{V}_5 = K_{st}\cdot\mathcal{P}^{(t)}_{\mathbf{0}}, & \mathcal{V}_6\,. = K_{st}\cdot\mathcal{P}^{(t)}_{\mathbf{27}}, \end{array} \tag{28}$$

and the eigenvalues with the corresponding Casimirs (18):

$$\{E_\kappa\} \;=\; \frac{1}{N}\,\{c_{\mathbf{a}},\, c_{\mathbf{s}},\, c_{\mathbf{10}},\, c_{\mathbf{1}},\, c_{\mathbf{0}},\, c_{\mathbf{27}}\}\,. \tag{29}$$

Since for $t \to 0$ scattering in the Born approximation is dominated by t-channel one gluon exchange, we are left with

$$S(\tau) \;=\; \chi_t(\tau) \;=\; \left(\frac{s}{t}\right)^{-2N\tau}, \tag{30}$$

which exponent coincides with the (twice) Regge trajectory of the gluon.

3 Final remark

The cubic equation (20) for the N-dependent energy levels 4, 5, 6 of $\mathcal{Q}$ possesses a weird symmetry which interchanges internal (colour group) and external (scattering angle) degrees of freedom:

$$\frac{T+U}{T-U} \quad\Longleftrightarrow\quad N\,. \tag{31}$$

In particular, this symmetry relates 90-degree scattering, $T=U$, with the large-N limit of the theory. Giving the complexity of the expressions involved, such a symmetry being accidental looks highly improbable. Its origin remains mysterious and may point at existence of an enveloping theoretical context that correlates internal and external variables (string theory?).

A technicalities

Soft anomalous dimension. The normalised anomalous dimension matrix $\mathcal{Q}$ has a block structure and reads

$$\mathcal{Q} \;=\; \begin{pmatrix} \frac{3}{2} & 0 & -2b & -\frac{1}{2}b & -\frac{2}{N^2}b & -\frac{2}{N^2}b \\ 0 & 1 & 0 & -b & -\frac{(N+1)(N-2)}{N^2}b & -\frac{(N-1)(N+2)}{N^2}b \\ -\frac{2}{N^2-1}b & 0 & 2 & 0 & 0 & 0 \\ -\frac{1}{2}b & -\frac{2}{N^2-4}b & 0 & \frac{3}{2} & 0 & 0 \\ -\frac{N+3}{2(N+1)}b & -\frac{N+3}{2(N+2)}b & 0 & 0 & \frac{N-1}{N} & 0 \\ -\frac{N-3}{2(N-1)}b & -\frac{N-3}{2(N-2)}b & 0 & 0 & 0 & \frac{N+1}{N} \end{pmatrix} \tag{32}$$

The matrix elements of the states **27** and **0** (two last rows and columns) are related by the formal operation $N \to -N$. Let us remark that our anomalous dimension differs from the one introduced in [3] by a piece proportional to the unit matrix. In our approach, this piece is absorbed into the collinear factor in (11) and determines the common hard scale of the Sudakov form factors in (9) as $Q^2 = tu/s$.

Colour re-projection matrices for $gg \to gg$. Graphic colour projector technique described in [7] allows one to find the re-projection matrices K_{ts} introduced in (12) without much effort The matrix K_{ts} is given by

$$K_{ts} = \begin{pmatrix} \frac{1}{2} & 0 & 1 & \frac{1}{2} & -\frac{1}{N} & \frac{1}{N} \\ 0 & \frac{1}{2} & \frac{N^2-4}{2} & -1 & -\frac{N-2}{2N} & -\frac{N+2}{2N} \\ \frac{1}{N^2-1} & \frac{1}{N^2-1} & \frac{1}{N^2-1} & \frac{1}{N^2-1} & \frac{1}{N^2-1} & \frac{1}{N^2-1} \\ \frac{1}{2} & -\frac{2}{N^2-4} & 1 & \frac{N^2-12}{2(N^2-4)} & \frac{1}{N+2} & -\frac{1}{N-2} \\ -\frac{N(N+3)}{4(N+1)} & \frac{-N(N+3)}{4(N+1)(N+2)} & \frac{N^2(N+3)}{4(N+1)} & \frac{N^2(N+3)}{4(N+1)(N+2)} & \frac{N^2+N+2}{4(N+1)(N+2)} & \frac{N+3}{4(N+1)} \\ \frac{N(N-3)}{4(N-1)} & \frac{-N(N-3)}{4(N-1)(N-2)} & \frac{N^2(N-3)}{4(N-1)} & \frac{-N^2(N-3)}{4(N-1)(N-2)} & \frac{N-3}{4(N-1)} & \frac{N^2-N+2}{4(N-1)(N-2)} \end{pmatrix} \tag{33}$$

The inverse matrix coincides with the direct one, $K_{st} = K_{ts}$. To construct the u-channel re-projection matrices K_{us} in (12) one exploits the symmetry of the s-channel projectors under $t \leftrightarrow u$ transformation. Thus K_{us} is obtained by changing sign of the first two *columns* of K_{ts} and the inverse matrix K_{su} — by changing sign of the first two *rows* of K_{ts}.

Eigenvectors. The eigenvectors $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3$ are

$$\mathcal{V}_{1,2,3} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \dfrac{4b}{N^2-4} \\ -\dfrac{b\,N(N+3)}{2(N+2)} \\ \dfrac{b\,N(N-3)}{2(N-2)} \end{bmatrix} \begin{bmatrix} 1+b \\ -2b \\ \dfrac{4b}{N^2-1} \\ 1+\dfrac{b\,(N^2-12)}{N^2-4} \\ -\dfrac{b\,N(N+3)}{(N+1)(N+2)} \\ -\dfrac{b\,N(N-3)}{(N-1)(N-2)} \end{bmatrix} \begin{bmatrix} -1+b \\ -2b \\ -\dfrac{4b}{N^2-1} \\ 1-\dfrac{b\,(N^2-12)}{N^2-4} \\ \dfrac{b\,N(N+3)}{(N+1)(N+2)} \\ \dfrac{b\,N(N-3)}{(N-1)(N-2)} \end{bmatrix} . \tag{34}$$

The states 2 and 3 are related by the crossing $t \leftrightarrow u$ transformation: $\mathcal{V}_3$ is obtained from $\mathcal{V}_2$ by $b \to -b$ and changing sign of the antisymmetric projectors (first two rows).

The last three eigenvectors are

$$\mathcal{V}_{4,5,6} = \begin{bmatrix} -\dfrac{4}{N^2}\,(E_i-1)\,b \\ -\dfrac{N^2-4}{N^2}\,(E_i-2)\,b \\ \dfrac{1}{N^2-1}\left[\left(E_i-\dfrac{N-1}{N}\right)\left(E_i-\dfrac{N+1}{N}\right)-\dfrac{N^2-5}{N^2}\,b^2\right] \\ \dfrac{4}{N^2}\,b^2 \\ \dfrac{N}{N+1}\left[\dfrac{N+2}{2\,N}(E_i-2)\left(E_i-\dfrac{N+1}{N}\right)-2\,b^2\right] \\ \dfrac{N}{N-1}\left[\dfrac{N-2}{2\,N}(E_i-2)\left(E_i-\dfrac{N-1}{N}\right)-2\,b^2\right] \end{bmatrix} , \tag{35}$$

with E_i the corresponding energy eigenvalue, $i = 4, 5, 6$.

Vectors $\mathcal{V}_i$ are orthogonal with respect to the scalar product defined by the metric tensor $W_{\alpha\beta} = K_\alpha \delta_{\alpha\beta}$,

$$\langle \mathcal{V}_i |\, W^{-1}\, | \mathcal{V}_k \rangle \;=\; 0\,, \quad i \neq k\,,$$

while the matrix $\mathcal{Q}$ becomes symmetric [7] under the corresponding metric transformation

$$W^{-1/2}\,\mathcal{Q}\,W^{1/2} \;=\; \mathcal{Q}_{sym}\,.$$

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